

The local load sharing fiber bundle model in higher dimensions

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We consider the local load sharing fiber bundle model in one to five dimensions. Depending on the breaking threshold distribution of the fibers, there is a transition where the fracture process becomes localized. In the localized phase, the model behaves as the invasion percolation model. The difference between the local load sharing fiber bundle model and the equal load sharing fiber bundle model decreases with increasing dimensionality as a power law.

The fiber bundle model has come a long way since its introduction in 1926 by Peirce [1]. Initially introduced to model the strength of yarn, the model has slowly gained ground as a fundamental model for fracture in somewhat the same way that the Ising model has become a paradigm for magnetic systems. In 1945, the presentation by Daniels [2] on the fiber bundle model as a statistical problem led to a continuous interest for the model in the mechanics community. The statistical physics community “discovered” the model in the early nineties in the aftermath of the surge of interest in fracture and breakdown phenomena in that community [3, 4].

The fiber bundle model introduced by Peirce [1] is today known as the *equal load sharing* (ELS) fiber bundle model. N Hookean springs — fibers — of length x_0 and spring constant κ are placed between two parallel infinitely stiff clamps. When the distance between the clamps is $x_0 + x$, each fiber carries a load $\sigma = \kappa x$. Each fiber i has a maximum elongation threshold x_i , upto which it can sustain before failing permanently. The threshold elongation is drawn from a probability density $p(x_i)$. The corresponding maximum load that fiber i can sustain is therefore $\sigma_i = \kappa x_i$. When a fiber fails, its load is shared equally among all the surviving fibers since the clamps are infinitely stiff — hence the expression “equal load sharing.”

The *local load sharing* (LLS) fiber bundle model was introduced by Harlow and Phoenix [5, 6] as a one-dimensional array of fibers, each having an independent breaking threshold drawn from some threshold distribution $p(x)$. They defined the force redistribution rule as follows: When a fiber fails, the load it carried is redistributed in equal portions onto its two nearest surviving neighbors. Hence, if a fiber i is adjacent to $n_{l,i}$ failed fibers to the left and $n_{r,i}$ failed fibers to the right, it will carry a load [7]

$$\sigma_i = \kappa \left[1 + \frac{n_{l,i} + n_{r,i}}{2} \right] x. \quad (1)$$

Whereas the ELS fiber bundle model is extreme in the sense that it redistributes the force carried by the failed fibers equally among all surviving fibers wherever they are placed, the local load sharing fiber bundle model is

extreme in the opposite sense: only the nearest survivors, pick up the force carried by the failed fiber. There are many models that are intermediate between the two extreme models. For example, the γ model of Hidalgo et al. [8] distributes the force carried by the failed fiber according to a power law in the distance from the failed fiber. The soft clamp model [9–12] replaces one of the infinitely stiff clamps in the ELS model by a clamp with finite elastic constant. Hence, the redistribution of the load of a failed fiber is governed by the elastic response of the soft clamp.

We emphasize the following subtle point in the implementation of the LLS model [4]. If the redistribution of forces after the failure of a fiber proceeds by dividing the force it carried in two and adding each half to the two nearest surviving fibers to the left and right — i.e., according to the recipe of Harlow and Phoenix [5] — the force distribution will not follow Eq. (1). Rather, it will become dependent on the order in which the fibers have failed. Hence, it will not be possible to determine the force distribution among the fibers only from the knowledge of present failed fibers in the system. This history dependency in the force distribution is unphysical. To give an example, if two adjacent fibers have failed and the two nearest surviving fibers each has one survivor, the first procedure will produce the following loads on the fibers: $(7/4, 0, 0, 9/4)$ or $(9/4, 0, 0, 7/4)$ depending on the order in which the two middle fibers were failed. According to Eq. (1), the load distribution should be $(2, 0, 0, 2)$ independent of the order in which the fibers failed.

When implementing the LLS model in two or more dimensions, the algorithm by which the forces are redistributed becomes even more crucial. We insist that the model should be physical where the force distribution among the surviving fibers can be determined by only knowing which fibers are already failed and it should not depend on the order in which they have failed. This leads to the concept of clusters of failed fibers, where the term “cluster” is used in the same sense as in the site percolation problem [13]: failed fibers that are nearest neighbors to each other form a cluster. The total load carried by all the failed fibers in a given cluster will then be shared equally by the surviving fibers that form the perimeter

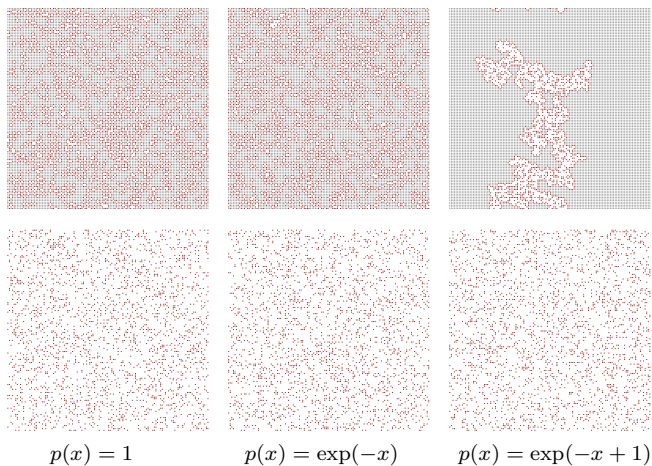


FIG. 1. (Color online) Snapshots of the two-dimensional LLS model after 1792 failed fibers (top row) and after 13824 failed fibers (bottom row). The first column shows a uniform threshold distribution on the unit interval, the middle column is an exponential threshold distribution $p(x) = \exp(-x)$ where $x \in [0, \infty)$. The third column shows the threshold distribution $p(x) = \exp(-x + 1)$ where $x \in [1, \infty)$. The system size is 128×128 . The red fibers are survivors adjacent to clusters of failed fibers, the gray fibers are survivors that are not adjacent to failed fibers and white fibers have failed.

to that cluster. If a surviving fiber is adjacent to two different clusters of failed fibers, the total load it carries is the sum of the loads contributed from both the clusters.

This generalization of the one-dimensional LLS model to higher dimensions is the simplest one that ensures history independence in the force distribution. A more elaborate generalization may be found in Patinet et al. [14]. Here, one of the clamps is exchanged for a stretchable membrane that has no bending resistance. The elastic response of this model is equivalent to the LLS model in one dimension. However, it differs from the one we propose here in two dimensions.

We show in Fig. 1 the two stages of the two-dimensional local load sharing model: after 1792 failed fibers (top row) and after 13824 failed fibers (bottom row). The total number of fibers was $N = 128^2$. The fibers, placed at the nodes of a square lattice, are seen from above. The failed fibers are shown as white, the intact fibers that belong to the external and internal perimeters of the clusters of failed fibers are shown as red. The intact fibers that do not belong to the perimeters are shown as gray. There are periodic boundary conditions in all directions. In the first column of the figure, the threshold distribution $p(x)$ was uniform on the unit interval. Hence, the cumulative probability was $P(x) = x$ where $x \in [0, 1]$. In the next two columns, the cumulative threshold probability was $P(x) = 1 - \exp(x_{<} - x)$ where $x \in [x_{<}, \infty)$. In the middle column, $x_{<} = 0$ and in the third column $x_{<} = 1$. In the top row, it is hard to distinguish the difference between the first two panels

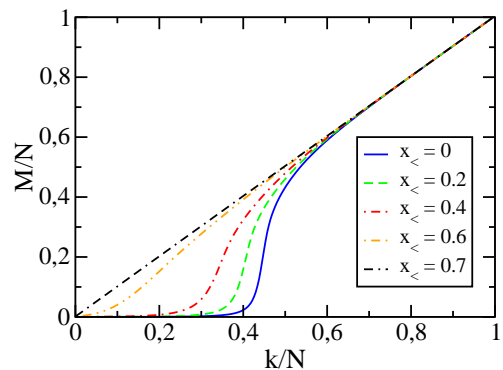


FIG. 2. (Color online) The size of the largest hole in the two-dimensional LLS model as a function of the relative number of failed fibers k/N , where $N = 256^2$. The threshold distribution was $p(x) = \exp(-x + x_{<})$, where $x \in [x_{<}, \infty)$. Each data set is based on 5000 samples.

of the figure. However, the third panel in the top row is very different. In this case, the breakdown process is *localized* from the very beginning. That is, a single cluster of failed fibers forms and keeps growing. On the other hand, three panels in the bottom row are all very similar.

When the breakdown process is localized so that only one cluster of failed fibers forms, the model is equivalent to the *invasion percolation model* [15]. In the invasion percolation model, each site is given a random number. An initial site is invaded. The perimeter of this one-site cluster form the growth sites and the growth site with the smallest random number associated with it is invaded. This is repeated, letting the perimeter of the cluster of invaded sites to be the growth sites. In the LLS model, the perimeter of the single cluster of failed fibers will carry the extra force that makes these and only these fibers liable for failure when the threshold is narrow enough to imply localization. It will be the fiber in the perimeter that has the smallest failure threshold that will fail next. Hence, it behaves precisely as the invasion percolation model.

The onset of localization is illustrated in Fig. 2. Here we show the size of the largest cluster of failed fibers, M as a function of the number of failed fibers k for different values of the lower cutoff $x_{<}$ in the threshold distribution $p(x) = \exp(x_{<} - x)$ where $x \in [x_{<}, \infty)$. When $x_{<} = 0.7$, the size of the largest cluster grows linearly with k from the very beginning, implying localization. On the other hand, when $x_{<} = 0$, the size of the largest cluster remains very small for a long period, and then grows rapidly afterwards. This is caused by merging of smaller clusters. However, the value at which the derivative of the curve is the largest is considerably smaller than the value $k/N \approx 0.592746$, the site percolation threshold [13], emphasizing that the failure process is *not* a random percolation process.

We now consider the breaking characteristics of the

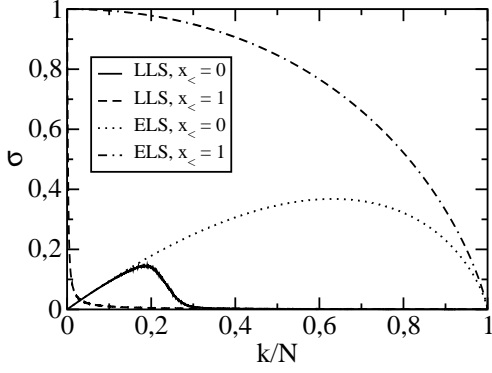


FIG. 3. Force per fiber σ as a function of the relative number of failed fibers k/N in the one-dimensional local load sharing model together with the ELS model result (Eq. 5). The thresholds were distributed according to $p(x) = \exp(-x + x_<)$ where $x \in [x_<, \infty)$ was used with $x_< = 0$ and 1 respectively. Here $N = 4000$. Each data series is based on 2000 samples.

LLS model in comparison to the ELS model. When k fibers have failed, the force F carried by the surviving fibers in the ELS model is

$$F = N\sigma = (N - k) \kappa x, \quad (2)$$

where we have defined the force per fiber $\sigma = F/N$. In the local load sharing model, we have

$$F = N\sigma = N \kappa x, \quad (3)$$

since the surviving perimeter fibers precisely absorb the load carried by the failed fibers.

We order the failure thresholds of the N fibers in an ascending sequence, $x_{(1)} < x_{(2)} < \dots < x_{(m)} < \dots < x_{(N)}$. According to order statistics [16], the average (over samples) of the m th member of this sequence is given by

$$P(\langle x_{(m)} \rangle) = \frac{m}{N} \quad (4)$$

for large N . We combine this equation with Eq. (2) for the ELS model assuming that $P(x) = 1 - \exp(-x + x_<)$ for $x \in [x_<, \infty)$ to find

$$\sigma = \left[1 - \frac{k}{N}\right] \left[x_< - \ln\left(1 - \frac{k}{N}\right)\right]. \quad (5)$$

For a uniform threshold distribution in $[x_<, 1]$, the cumulative probability is $P(x) = (x - x_<)/(1 - x_<)$ we find

$$\sigma = \left[1 - \frac{k}{N}\right] \left[x_< + (1 - x_<) \frac{k}{N}\right]. \quad (6)$$

We show the ELS behavior for the exponential threshold distribution (Eq. 5) in Fig. 3 together with the corresponding curves ($x_< = 0$ and $x_< = 1$) for the LLS model in one dimension. There is a large difference between ELS and LLS models.

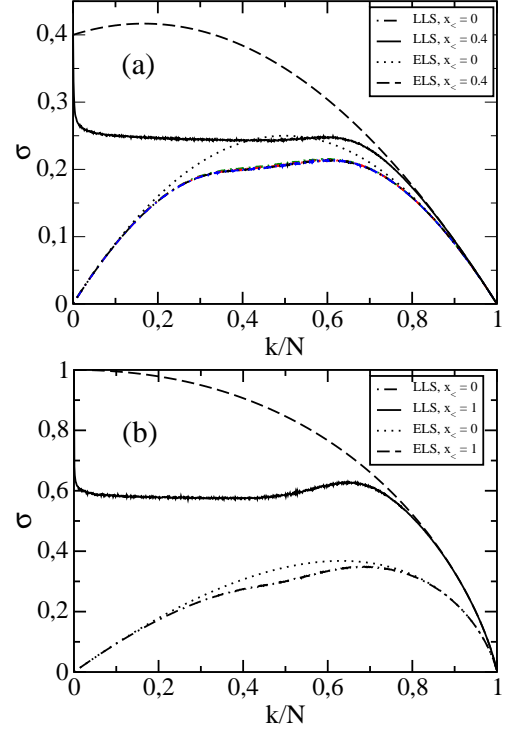


FIG. 4. (Color online) Force per fiber σ as a function of the relative number of failed fibers k/N in the two-dimensional LLS model compared with the ELS model result. The threshold distribution in (a) was uniform in the interval $x \in [x_<, 1]$ with $x_< = 0$ and 0.4. In (b), the distribution $p(x) = \exp(x_< - x)$ where $x \in [x_<, \infty)$ with $x_< = 0$ and 1 respectively. In (a) for LLS, results are also shown for different system sizes, $N = 32^2$ (green), 64^2 (red), 128^2 (black), and 256^2 (blue) for the uniform distribution with $x_< = 0$. For the other plots, $N = 256^2$. Each data series is based on 5000 samples.

This picture changes in two dimensions. In Fig. 4, we show the results for the two-dimensional LLS model for uniform threshold distribution with cumulative threshold probabilities $P(x) = (x - x_<)/(1 - x_<)$, where $x \in [x_<, 1]$ with $x_< = 0$ and 0.4 in (a). In Fig. 4(b), $P(x) = 1 - \exp(x_< - x)$ where $x \in [x_<, \infty)$ with $x_< = 0$ and 1. When comparing this figure to the corresponding one for one dimension (Fig. 3), we see that the LLS model now is much closer to the ELS model than in one dimension.

It should be pointed out that σ vs. k/N for the exponential threshold distribution with $x_< = 1$ has a curious upwards bend before its maximum value, see Fig. 4. A small upwards bend can also be seen for the uniform threshold distribution with $x_< = 0.4$. This means that the model is stable in this region in the sense that if σ is used as the control parameter, fiber failures will only occur if σ is increased. This is not true in the ELS model. Hence, the LLS model is in fact *more stable* than the ELS model in this region.

The similarity between the ELS and LLS models is also

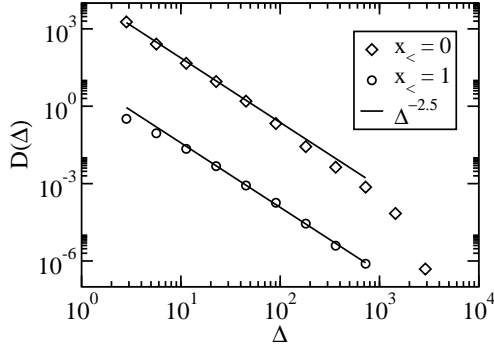


FIG. 5. The burst distribution in the two-dimensional LLS model. The threshold distribution was $p(x) = \exp(x_{<} - x)$ where $x \in [x_{<}, \infty)$. The data sets are based on 5000 samples of size $N = 256^2$.

evident in other quantities that characterize the two models. In Fig. 5 we show the burst distribution for the LLS model in two dimensions for the cumulative threshold probability $P(x) = 1 - \exp(x_{<} - x)$ where $x \in [x_{<}, \infty)$ with $x_{<} = 0$ and 1. The burst distribution is the histogram of the number of simultaneously failing fibers Δ when the force σ is the control parameter. Hemmer and Hansen showed in 1992 that the burst distribution in the ELS model is given by

$$\omega(\Delta) \sim \Delta^{-5/2}, \quad (7)$$

for a very wide class of threshold distributions to which $p(x) = \exp(x_{<} - x)$ belongs [17].

Later, Hansen and Hemmer investigated the burst distribution in the one-dimensional LLS model finding a burst exponent ≈ 4.5 rather than $5/2$ [18]. Kloster *et al.* [7] showed analytically that the burst distribution falls off faster than a power law in the LLS model when the threshold distribution is uniform on the unit interval. Fig. 5 shows that the burst distribution in the two-dimensional LLS model is consistent with Eq. (7) for both $x_{<} = 0$ and $x_{<} = 1$.

In Fig. 6, we show the σ vs. k/N curves for the *three-dimensional*, *four-dimensional* and *five-dimensional* LLS fiber bundle model for the cumulative threshold probability $P(x) = x$ with $x \in [0, 1]$ (top row) and $P(x) = 1 - \exp(x_{<} - x)$ with $x \in [x_{<}, \infty)$ (bottom row). We compare the curves with the ELS model results given in Eqs. (5) and (6). Interestingly, the curves for the local and the ELS models are approaching each other more and more as the dimensionality is increased. The difference in σ for LLS and ELS for different system dimensions is measured and plotted in Fig. 7 for the two threshold distributions.

It can be noticed that the maxima of the $\Delta\sigma$ curves shifts towards smaller k/N with changing dimensionality. Therefore, in order to quantify the difference between LLS and ELS models, we measure the total area ($\Delta\sigma_{\text{area}}$)

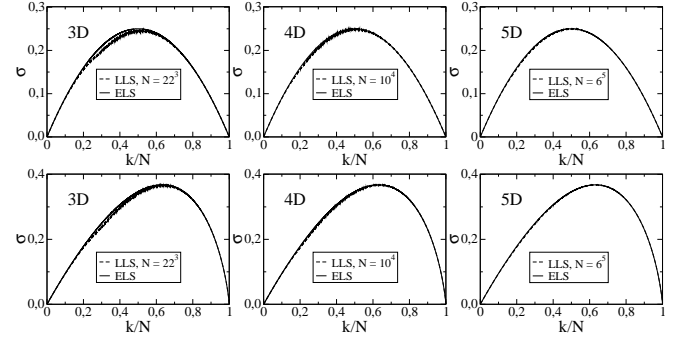


FIG. 6. Force per fiber σ as a function of the relative number of failed fibers k/N in the three-dimensional (3D), four-dimensional (4D) and five-dimensional (5D) LLS model. The top row corresponds to the threshold distributions $P(x) = x$ with $x \in [0, 1]$ and bottom row corresponds to $P(x) = 1 - \exp(x_{<} - x)$ with $x \in [x_{<}, \infty)$. The system sizes are indicated in the figures. The number of samples over which the data are averaged are 80000, 40000 and 40000 for three, four and five dimensions respectively.

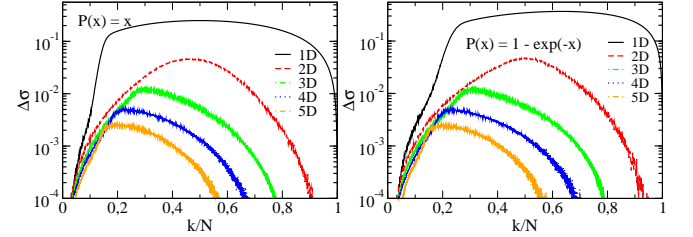


FIG. 7. (Color online) Difference of the force per fiber σ in the LLS model from that in ELS model ($\Delta\sigma$) for two threshold distributions $P(x) = x$ with $x \in [0, 1]$ and $P(x) = 1 - \exp(x_{<} - x)$ with $x \in [x_{<}, \infty)$ in one, two, three, four and five dimensions. A rapid decrease in $\Delta\sigma$ can be observed with increasing dimensionality.

under the $\Delta\sigma$ curves. In Fig. 8, we plot $\Delta\sigma_{\text{area}}$ as a function of the dimensionality D of the system. Interestingly, a power-law dependency

$$\Delta\sigma_{\text{area}} \sim D^{-\mu}, \quad (8)$$

with $\mu = 3.5 \pm 0.1$ is observed.

In the cases where the threshold cutoff is $x_{<} > 0$, there is a non-negligible N -dependency in the σ vs. k/N curves and the effective exponent μ needs further finite size scaling analysis to be determined.

From Eq. (8) we conclude that there is no finite upper critical dimension for which the LLS and ELS models become equal. However, the difference falls off rapidly with increasing D .

Finally, we like to highlight about the breaking process which makes the LLS and the ELS models similar at the earlier and later of the breakdown process when there is no localization. The right column in Fig. 1 shows the two-dimensional LLS model after 13824 out of $128^2 = 16384$ fibers in total have failed, $k/N \approx 0.84$. The clusters of

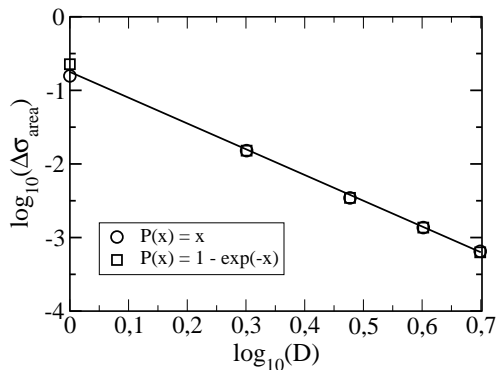


FIG. 8. The area under the $\Delta\sigma$ curves in Fig. 7 as a function of dimensionality D .

failed fibers have merged and essentially all the remaining fibers have become part of the perimeter of a single percolating cluster of failed fibers. Hence, all the remaining fibers experience the same force as they all are adjacent to the same cluster — and hence, they all share the same force as in the ELS fiber bundle model.

Early in the breakdown process, when there is no localization, fibers will fail not due to being under stress because they are on the perimeter of clusters of already failed fibers, but because they have small thresholds. Hence, early in the breakdown process, we expect the LLS and the ELS models to be quite similar. This is true in all dimensions, except when there is localization, see Figs. 3, 4 and 6.

The LLS model is extreme in that it is the perimeter fibers that absorb the forces from the failed fibers. We have mentioned models that are in between the ELS and the LLS models. When the LLS and ELS models are rapidly approaching each other with increasing D , so will the in-between models also; they will rapidly approach the ELS model with increasing D . This argument also apply to models that normally are not classified as fiber bundle models, such as the fuse model where Zapperi *et al.* [19] has reported a burst distribution exponent in three dimensions equal to 2.55, close to the ELS value 5/2. Hence, already in three dimensions, the ELS model is not far from the much more complex models of fracture.

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- [1] F. T. Peirce, J. Text. Ind. **17**, 355 (1926).
- [2] H. E. Daniels, Proc. Roy. Soc. London **A183**, 405 (1945).
- [3] S. Pradhan, A. Hansen and B. K. Chakrabarti, Rev. Mod. Phys. **82**, 499 (2010).
- [4] A. Hansen, P. C. Hemmer and S. Pradhan, *The fiber bundle: modeling failure in materials* (J. Wiley, Chichester, 2015).
- [5] D. G. Harlow and S. L. Phoenix, J. Composite Mater. **12**, 195 (1978).
- [6] D. G. Harlow and S. L. Phoenix, J. Mech. Phys. Solids **39**, 173 (1991).
- [7] M. Kloster, A. Hansen and P. C. Hemmer, Phys. Rev. E **56**, 2615 (1997).
- [8] R. C. Hidalgo, Y. Moreno, F. Kun and H. J. Herrmann, Phys. Rev. E **65**, 046148 (2002).
- [9] G. G. Batrouni, A. Hansen and J. Schmittbuhl, Phys. Rev. E **65**, 036126 (2002).
- [10] A. Stormo, K. S. Gjerden and A. Hansen, Phys. Rev. E **86**, R025101 (2012).
- [11] K. S. Gjerden, A. Stormo and A. Hansen, Phys. Rev. Lett. **111**, 135502 (2013).
- [12] K. S. Gjerden, A. Stormo and A. Hansen, Front. Phys. **2**, 66 (2014).
- [13] D. Stauffer and A. Aharony, *Introduction to percolation theory* (Taylor and Francis, London, 1994).
- [14] S. Patinet, D. Vandembroucq, A. Hansen and S. Roux, Europ. J. Phys. Spec. Top. **223**, 2339 (2014).
- [15] D. Wilkinson and J. F. Willemsen, J. Phys. A **16**, 3365 (1983).
- [16] E. J. Gumbel, *Statistics of extremes* (Dover Publ., Mineola New York, 2004).
- [17] P. C. Hemmer and A. Hansen, ASME J. Appl. Mech. **59**, 909 (1992).
- [18] A. Hansen and P. C. Hemmer, Phys. Lett. A **184**, 394 (1994).
- [19] S. Zapperi, P. K. V. V. Nukala and S. Šimunović, Physica A **357**, 129 (2005).